

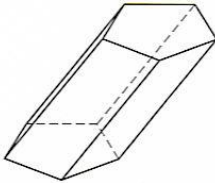
The Geometry of Solids

In this lesson you will

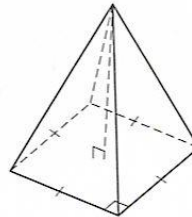
- Learn about **polyhedrons**, including **prisms** and **pyramids**
- Learn about solids with curved surfaces, including **cylinders**, **cones**, and **spheres**

In this chapter you will study three-dimensional solid figures. Lesson 10.1 introduces various types of three-dimensional solids and the terminology associated with them. Read the lesson. Then review what you read by completing the statements and answering the questions below.

1. A polyhedron is a solid formed by _____ that enclose a single region of space.
2. A segment where two faces of a polyhedron intersect is called a(n) _____.
3. A polyhedron with six faces is called a(n) _____.
4. A tetrahedron has _____ faces.
5. If each face of a polyhedron is enclosed by a regular polygon, and each face is congruent to the other faces, and the faces meet each vertex in exactly the same way, then the polyhedron is called a(n) _____.
6. A(n) _____ is a polyhedron with two faces, called bases, that are congruent, parallel polygons.
7. The faces of a prism that are not bases are called _____.
8. What is the difference between a right prism and an oblique prism?
9. What type of solid is shown below?



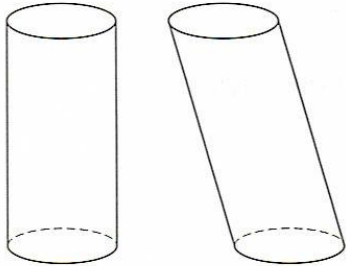
10. How many bases does a pyramid have?
11. The point that all the lateral faces of a pyramid have in common is the _____ of the pyramid.
12. What is the difference between an altitude of a pyramid and the height of a pyramid?
13. What type of solid is shown at right?



(continued)

Lesson 10.1 • The Geometry of Solids (continued)

14. Name three types of solids with curved surfaces.
15. A(n) _____ is the set of all points in space at a given distance from a given point.
16. A circle that encloses the base of a hemisphere is called a(n) _____.
17. Give an example of a real object that is shaped like a cylinder. Explain how you know it is a cylinder.
18. Tell which cylinder below is an oblique cylinder and which is a right cylinder. For each cylinder, draw and label the *axis* and the *altitude*.



19. The base of a cone is a(n) _____.
20. If the line segment connecting the vertex of a cone with the center of the base is perpendicular to the base, then the cone is a(n) _____.

After you have finished, check your answers below.

- | | |
|--|---|
| <ol style="list-style-type: none">1. polygons2. edge3. hexahedron4. four5. regular polygon6. prism7. lateral faces8. In a right prism all lateral faces are rectangles and are perpendicular to the base. In an oblique prism they are not.9. pentagonal prism (it may be oblique or right)10. one11. vertex12. The altitude is a segment from the vertex of the pyramid to the plane of the base that is perpendicular to the plane of the base. The height is the length of the altitude. | <ol style="list-style-type: none">13. square pyramid14. possible answers: cylinder, cone, sphere, hemisphere15. sphere16. great circle17. The example will vary. Reason: It has two parallel bases and both bases are circles.18. The figure on the left is a right cylinder, and the figure on the right is an oblique cylinder. See page 523 of the book for labeled pictures of both.19. circle20. right cone |
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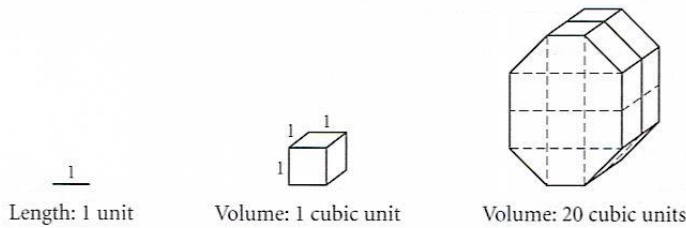
10.2

Volume of Prisms and Cylinders

In this lesson you will

- Discover the volume formula for **right rectangular prisms**
- Extend the volume formula to **right prisms** and **right cylinders**
- Extend the volume formula to **oblique prisms** and **oblique cylinders**

Volume is the measure of the amount of space contained in a solid. You use cubic units to measure volume: cubic inches (in^3), cubic feet (ft^3), cubic yards (yd^3), cubic centimeters (cm^3), cubic meters (m^3), and so on. The volume of an object is the number of unit cubes that completely fill the space within the object.



Investigation: The Volume Formula for Prisms and Cylinders

Read Step 1 of the investigation in your book.

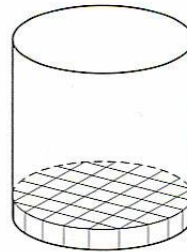
Notice that the number of cubes in the base layer is the number of square units in the area of the base and that the number of layers is the height of the prism. So, you can use the area of the base and the height to find the volume of a right rectangular prism.

Rectangular Prism Volume Conjecture If B is the area of the base of a right rectangular prism and H is the height of the solid, then the formula for the volume is $V =$ _____.

C-86a

You can find the volume of any other right prism or cylinder the same way—by multiplying the area of the base by the height. For example, to find the volume of this cylinder, find the area of the circular base (the number of cubes in the base layer) and multiply by the height.

You can extend the Rectangular Prism Volume Conjecture to all right prisms and right cylinders. Complete the conjecture below.



Right Prism-Cylinder Volume Conjecture If B is the area of the base of a right prism (or cylinder) and H is the height of the solid, then the formula for the volume is $V =$ _____.

C-86b

(continued)

Lesson 10.2 • Volume of Prisms and Cylinders (continued)

What about the volume of an *oblique* prism or cylinder? Page 532 of your book shows that you can model an oblique rectangular prism with a slanted stack of paper. You can then “straighten” the stack to form a right rectangular prism with the same volume. The right prism has the same base area and the same height as the oblique prism. So, you find the volume of the oblique prism by multiplying the area of its base by its height. A similar argument works for other oblique prisms and cylinders.

Now you can extend the Right Prism-Cylinder Volume Conjecture to oblique prisms and cylinders.

Oblique Prism-Cylinder Volume Conjecture The volume of an oblique prism (or cylinder) is the same as the volume of a right prism (or cylinder) that has the same _____ and the same _____.

C-86c

Be careful when calculating the volume of an oblique prism or cylinder. Because the lateral edges are not perpendicular to the bases, the height of the prism or cylinder is *not* the length of a lateral edge.

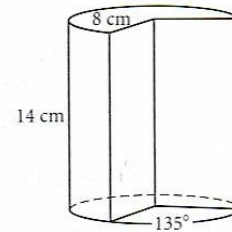
You can combine the last three conjectures into one conjecture.

Prism-Cylinder Volume Conjecture The volume of a prism or a cylinder is the _____ multiplied by the _____.

C-86

Examples A and B in your book show you how to find the volumes of a trapezoidal prism and an oblique cylinder. Read both examples. Try to find the volumes yourself before reading the solutions. Then read the example below.

EXAMPLE The solid at right is a right cylinder with a 135° slice removed. Find the volume of the solid. Round your answer to the nearest cm.



► **Solution** The base is $\frac{225}{360}$, or $\frac{5}{8}$, of a circle with radius 8 cm. The whole circle has area 64π cm², so the base has area $\frac{5}{8}(64\pi)$, or 40π cm². Now use the volume formula.

$$\begin{aligned} V &= BH && \text{The volume formula.} \\ &= 40\pi(14) && \text{Substitute the area of the base and the height.} \\ &= 560\pi && \text{Multiply.} \\ &\approx 1759 && \text{Use the } \pi \text{ key on your calculator to get an approximate answer.} \end{aligned}$$

The volume is about 1759 cm³.

Volume of Pyramids and Cones

In this lesson you will

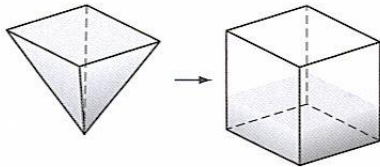
- Discover the volume formula for pyramids and cones

There is a simple relationship between the volumes of prisms and pyramids with congruent bases and the same height, and between cylinders and cones with congruent bases and the same height.

Investigation: The Volume Formula for Pyramids and Cones

If you have the materials listed at the beginning of the investigation, follow the steps in your book before continuing. The text below summarizes the results of the investigation.

Suppose you start with a pyramid and a prism with congruent bases and the same height. If you fill the pyramid with sand or water and then pour the contents into the prism three times, you will exactly fill the prism. In other words, the volume of the pyramid is $\frac{1}{3}$ the volume of the prism.



You will get the same result no matter what shape the bases have, as long as the base of the pyramid is congruent to the base of the prism and the heights are the same.

If you repeat the experiment with a cone and a cylinder with congruent bases and the same height, you will get the same result. That is, you can empty the contents of the cone into the cylinder exactly three times.

The results can be summarized in a conjecture.

Pyramid-Cone Volume Conjecture If B is the area of the base of a pyramid or a cone and H is the height of the solid, then the formula for the volume is $V = \frac{1}{3}BH$.

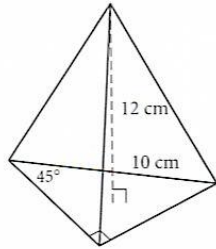
C-87

Example A in your book shows how to find the volume of a regular hexagonal pyramid. In the example, you need to use the 30° - 60° - 90° Triangle Conjecture to find the apothem of the base. Example B involves the volume of a cone. Work through both examples. Then read the example on the next page.

(continued)

Lesson 10.3 • Volume of Pyramids and Cones (continued)

EXAMPLE Find the volume of this triangular pyramid.



► **Solution** The base is an isosceles right triangle. To find the area of the base, you need to know the lengths of the legs. Let l be the length of a leg, and use the Isosceles Right Triangle Conjecture.

$$l(\sqrt{2}) = 10 \quad \text{The length of the hypotenuse is the length of a leg times } \sqrt{2}.$$

$$l = \frac{10}{\sqrt{2}} \quad \text{Solve for } l.$$

The length of each leg is $\frac{10}{\sqrt{2}}$. Now find the area of the triangle.

$$A = \frac{1}{2}bh \quad \text{Area formula for triangles.}$$

$$= \frac{1}{2}\left(\frac{10}{\sqrt{2}}\right)\left(\frac{10}{\sqrt{2}}\right) \quad \text{Substitute the known values.}$$

$$= 25 \quad \text{Multiply.}$$

So, the base has area 25 cm^2 . Now find the volume of the pyramid.

$$V = \frac{1}{3}BH \quad \text{Volume formula for pyramids and cones.}$$

$$= \frac{1}{3}(25)(12) \quad \text{Substitute the known values.}$$

$$= 100 \quad \text{Multiply.}$$

The volume of the pyramid is 100 cm^3 .

10.4

Volume Problems

In this lesson you will

- Use the volume formulas you have learned to solve problems

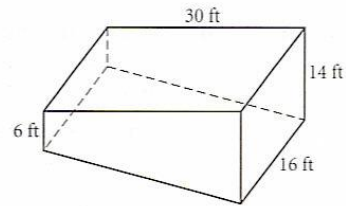
You have learned volume formulas for prisms, cylinders, pyramids, and cones.

In this lesson you will use these formulas to solve problems.

In Example A in your book, the volume of a right triangular prism is given and you must find the height. In Example B, the volume of a sector of a right cylinder is given and you must find the radius of the base. Try to solve the problems yourself before reading the solutions. Below are some more examples.

EXAMPLE A

A swimming pool is in the shape of the prism shown at right. How many gallons of water can the pool hold? (A cubic foot of water is about 7.5 gallons.)

► **Solution**

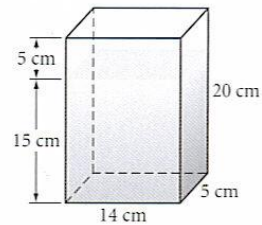
First, find the volume of the pool in cubic feet. The pool is in the shape of a trapezoidal prism. The trapezoid has bases of length 6 feet and 14 feet and a height of 30 feet. The height of the prism is 16 feet.

$$\begin{aligned} V &= BH && \text{Volume formula for prisms.} \\ &= \frac{1}{2}(30)(6 + 14) \cdot 16 && \text{Use the formula } \frac{1}{2}h(b_1 + b_2) \text{ for the area of a trapezoid.} \\ &= 4800 && \text{Solve.} \end{aligned}$$

The pool has volume 4800 ft³. A cubic foot is about 7.5 gallons, so the pool holds 4800(7.5), or 36,000 gallons of water.

EXAMPLE B

A sealed rectangular container 5 cm by 14 cm by 20 cm is sitting on its smallest face. It is filled with water up to 5 cm from the top. How many centimeters from the bottom will the water level reach if the container is placed on its largest face?

► **Solution**

The smallest face is a 5-by-14-centimeter rectangle. When the prism is resting on its smallest face, the water is in the shape of a rectangular prism with base area 70 cm² and height 15 cm. So, the volume of the water is 1050 cm³.

(continued)

Lesson 10.4 • Volume Problems (continued)

If the container is placed on its largest face, the volume will still be 1050 cm^3 , but the base area and height will change. The area of the new base will be $14(20)$, or 280 cm^2 . You can use the volume formula to find the height.

$$V = BH \quad \text{Volume formula for prisms.}$$

$$1050 = 280H \quad \text{Substitute the known values.}$$

$$3.75 = H \quad \text{Solve for } H.$$

The height of the water will be 3.75 cm. So, the water level will be 3.75 cm from the bottom of the container.

EXAMPLE C Find the volume of a rectangular prism with dimensions that are twice those of another rectangular prism with volume 120 cm^3 .

► **Solution** For the rectangular prism with volume 120 cm^3 , let the dimensions of the rectangular base be x and y and the height be z . The volume of this prism is xyz , so $xyz = 120$.

The dimensions of the base of the other prism are $2x$ and $2y$, and the height is $2z$. Let V be the volume of this prism. Then

$$V = BH \quad \text{Volume formula for prisms.}$$

$$= (2x)(2y)(2z) \quad \text{Substitute the known values.}$$

$$= 8xyz \quad \text{Multiply.}$$

$$= 8(120) \quad \text{Substitute the known value.}$$

$$= 960 \quad \text{Multiply.}$$

The volume of the larger prism is 960 cm^3 .

Displacement and Density

In this lesson you will

- Learn how the idea of **displacement** can be used to find the volume of an object
- Learn how to calculate the **density** of an object

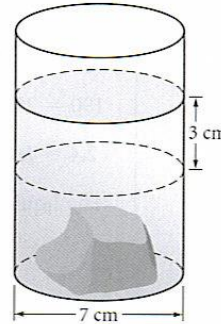
To find the volumes of geometric solids, such as prisms and cones, you can use a volume formula. But what if you want to find the volume of an irregularly shaped object like a rock? As Example A in your book illustrates, you can submerge the object in a regularly shaped container filled with water. The volume of water that is **displaced** will be the same as the volume of the object. Read Example A in your book. Then read the example below.

EXAMPLE A When Tom puts a rock into a cylindrical container with diameter 7 cm, the water level rises 3 cm. What is the volume of the rock to the nearest tenth of a cubic centimeter?

► **Solution** The “slice” of water that is displaced is a cylinder with diameter 7 cm and height 3 cm. Use the volume formula to find the volume of the displaced water.

$$\begin{aligned} V &= BH \\ &= \pi(3.5)^2 \cdot 3 \\ &\approx 115.5 \end{aligned}$$

The volume of the rock is about 115.5 cm^3 , the same as the volume of the displaced water.



An important property of a material is its density. **Density** is the mass of matter in a given volume. It is calculated by dividing mass by volume.

$$\text{density} = \frac{\text{mass}}{\text{volume}}$$

The table on page 551 of your book gives the densities of ten metals. In Example B, the mass of a clump of metal and information about the amount of water it displaces are used to identify the type of metal. Read this example carefully. Then read the following example.

(continued)

Lesson 10.5 • Displacement and Density (continued)

EXAMPLE B

Chemist Preethi Bhatt is given a clump of metal and is told that it is sodium. She finds that the metal has a mass of 184.3 g. She places it into a nonreactive liquid in a cylindrical beaker with base diameter 10 cm. If the metal is indeed sodium, how high should the liquid level rise?

► **Solution**

The table on page 551 of your book indicates that the density of sodium is 0.97 g/cm^3 . Use the density formula to find what the volume of the clump of metal should be if it is sodium.

$$\text{density} = \frac{\text{mass}}{\text{volume}} \quad \text{Density formula.}$$

$$0.97 = \frac{184.3}{\text{volume}} \quad \text{Substitute the known information.}$$

$$\text{volume} \cdot 0.97 = 184.3 \quad \text{Multiply both sides by } \textit{volume}.$$

$$\text{volume} = \frac{184.3}{0.97} \quad \text{Divide both sides by } 0.97.$$

$$\text{volume} = 190 \quad \text{Simplify.}$$

So, if the metal is sodium, it should displace 190 cm^3 of water. Use the volume formula to find the height of the liquid that should be displaced.

$$V = BH \quad \text{Volume formula for cylinders.}$$

$$190 = (5^2\pi)H \quad \text{Substitute the known information. (The base is a circle with radius 5 cm.)}$$

$$190 = 25\pi H \quad \text{Multiply.}$$

$$2.4 \approx H \quad \text{Solve for } H.$$

The liquid should rise about 2.4 cm.

10.6

Volume of a Sphere

In this lesson you will

- Discover the volume formula for a sphere

You can find the volume of a sphere by comparing it to the volume of a cylinder. In the investigation you will see how.

Investigation: The Formula for the Volume of a Sphere

If you have the materials listed at the beginning of the investigation, follow the steps in your book before continuing. The text below summarizes the results of the investigation.

Suppose you have a hemisphere and a cylinder. The radius of the cylinder equals the radius of the hemisphere, and the height of the cylinder is twice the radius. Note that the cylinder is the smallest one that would enclose a sphere made from two of the hemispheres.

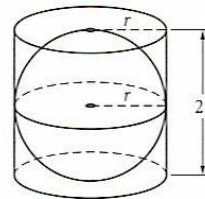
The volume of the cylinder is $\pi r^2(2r)$, or $2\pi r^3$.

If you fill the hemisphere with sand or water and empty the contents into the cylinder, the cylinder will be $\frac{1}{3}$ full.

If you fill the hemisphere again and empty the contents into the cylinder, the cylinder will be $\frac{2}{3}$ full. So, the volume of the sphere (two hemispheres) is equal to $\frac{2}{3}$ the volume of the cylinder.

The volume of the cylinder is $2\pi r^3$. So, the volume of the sphere is $\frac{2}{3}(2\pi r^3)$, or $\frac{4}{3}\pi r^3$.

The results can be summarized as a conjecture.



Sphere Volume Conjecture The volume of a sphere with radius r is given by the formula $V = \frac{4}{3}\pi r^3$.

C-88

Read Example A in your book, which involves finding the percentage of plaster cut away when the largest possible sphere is carved from a cube. The solution involves four steps:

- Step 1** Find the volume of the sphere.
- Step 2** Find the volume of the cube.
- Step 3** Subtract the volume of the sphere from the volume of the cube to find the volume of the plaster cut away.
- Step 4** Divide the volume cut away by the volume of the original cube and convert the answer to a percent to find the percentage cut away.

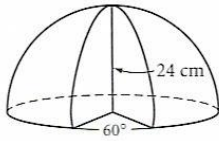
Read Example B in your book. Solve the problem yourself and then check your work by reading the given solution.

The following two examples give you more practice working with the volume of a sphere.

(continued)

Lesson 10.6 • Volume of a Sphere (continued)

EXAMPLE A Find the volume of this solid.



► **Solution** The solid is a hemisphere with a 60° sector cut away. First, find the volume of the entire hemisphere. Because the formula for the volume of a sphere is $V = \frac{4}{3}\pi r^3$, the formula for the volume of a hemisphere is $V = \frac{2}{3}\pi r^3$.

$$\begin{aligned} V &= \frac{2}{3}\pi r^3 && \text{Volume formula for a hemisphere.} \\ &= \frac{2}{3}\pi(24)^3 && \text{The radius is 24 cm.} \\ &= 9216\pi && \text{Simplify.} \end{aligned}$$

The volume of the entire hemisphere is $9216\pi \text{ cm}^3$. A 60° sector has been cut away, so the fraction of the hemisphere that remains is $\frac{300}{360}$, or $\frac{5}{6}$. So, the volume of the solid is $\frac{5}{6}(9216\pi) = 7680\pi \text{ cm}^3$, or about $24,127 \text{ cm}^3$.

EXAMPLE B A marble is submerged in water in a rectangular prism with a 2 cm-by-2 cm base. The water in the prism rises 0.9 cm when the marble is submerged. What is the diameter of the marble?

► **Solution** First find the volume of the water displaced by the marble.

$$\begin{aligned} V &= BH && \text{Volume formula for a prism.} \\ &= (2)(2)(0.9) && \text{Substitute the known information.} \\ &= 3.6 && \text{Simplify.} \end{aligned}$$

So, the volume of the displaced water, and thus of the marble, is 3.6 cm^3 .

Next use the volume of the marble to find its radius. Substitute 3.6 for V in the formula for the volume of a sphere and solve for r .

$$\begin{aligned} V &= \frac{4}{3}\pi r^3 && \text{Volume formula for a sphere.} \\ 3.6 &= \frac{4}{3}\pi r^3 && \text{Substitute the known information.} \\ \frac{\frac{3}{4}(3.6)}{\pi} &= r^3 && \text{Multiply both sides by } \frac{3}{4} \text{ and divide by } \pi. \\ 0.86 &\approx r^3 && \text{Simplify.} \\ r &\approx \sqrt[3]{0.86} \approx 0.95 && \text{Take the cube root of both sides.} \end{aligned}$$

The radius of the marble is about 0.95 cm, so the diameter is about 1.9 cm.

Surface Area of a Sphere

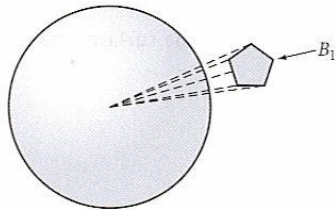
In this lesson you will

- Discover the formula for the **surface area of a sphere**

You can use the formula for the volume of a sphere, $V = \frac{4}{3}\pi r^3$, to find the formula for the surface area of a sphere.

Investigation: The Formula for the Surface Area of a Sphere

Imagine a sphere's surface divided into tiny shapes that are nearly flat. The surface area of the sphere is equal to the sum of the areas of these "near polygons." If you imagine radii connecting each of the vertices of the "near polygons" to the center of the sphere, you are mentally dividing the volume of the sphere into many "near pyramids." Each of the "near polygons" is a base for one of the pyramids, and the radius, r , of the sphere is the height of the pyramid. The volume, V , of the sphere is the sum of the volumes of all the pyramids.



Step 1 Imagine that the surface of a sphere is divided into 1000 "near polygons" with areas $B_1, B_2, B_3, \dots, B_{1000}$. The surface area, S , of the sphere is the sum of the areas of these "near polygons":

$$S = B_1 + B_2 + B_3 + \dots + B_{1000}$$

Step 2 The pyramid with base area B_1 has volume $\frac{1}{3}(B_1)r$, the pyramid with base area B_2 has volume $\frac{1}{3}(B_2)r$, and so on. The volume of the sphere, V , is the sum of these volumes:

$$V = \frac{1}{3}(B_1)r + \frac{1}{3}(B_2)r + \frac{1}{3}(B_3)r + \dots + \frac{1}{3}(B_{1000})r$$

Factor $\frac{1}{3}r$ from each term on the right side of the equation:

$$V = \frac{1}{3}r(B_1 + B_2 + B_3 + \dots + B_{1000})$$

Step 3 Because $V = \frac{4}{3}\pi r^3$, you can substitute $\frac{4}{3}\pi r^3$ for V :

$$\frac{4}{3}\pi r^3 = \frac{1}{3}r(B_1 + B_2 + B_3 + \dots + B_{1000})$$

Now substitute S for $B_1 + B_2 + B_3 + \dots + B_{1000}$:

$$\frac{4}{3}\pi r^3 = \frac{1}{3}rS$$

(continued)

Lesson 10.7 • Surface Area of a Sphere (continued)

Step 4 Solve the last equation for the surface area, S .

$$\frac{4}{3}\pi r^3 = \frac{1}{3}rS \quad \text{The equation from Step 3.}$$

$$4\pi r^3 = rS \quad \text{Multiply both sides by 3.}$$

$$4\pi r^2 = S \quad \text{Divide both sides by } r.$$

You now have a formula for finding the surface area of a sphere in terms of the radius. You can state the result as a conjecture.

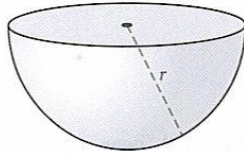
Sphere Surface Area Conjecture The surface area, S , of a sphere with radius r is given by the formula $S = 4\pi r^2$.

C-89

The example in your book shows how to find the surface area of a sphere if you know its volume. Try to find the surface area on your own before reading the solution. Then solve the problem in the example below.

EXAMPLE

The base of this hemisphere has circumference 32π cm. Find the surface area of the hemisphere (including the base).



► Solution

Because the base of the hemisphere has circumference 32π cm, the radius is 16 cm.

The area of the base of the hemisphere is $\pi(16)^2$, or 256π cm².

The area of the curved surface is half the surface area of a sphere with radius 16 cm.

$$\begin{aligned} S &= \frac{1}{2} \cdot 4\pi r^2 \\ &= \frac{1}{2} \cdot 4\pi(16)^2 \\ &= 512\pi \end{aligned}$$

So, the total surface area is $256\pi + 512\pi = 768\pi$ cm², or about 2413 cm².